A NEW CLASS OF TOPOLOGICAL SPACES

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF ALBERTA

EDMONTON, ALBERTA

AUGUST, 1966

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "A NEW CLASS OF TOPOLOGICAL SPACES", submitted by PAUL A. O'MEARA in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

A pseudobase $\mathcal{P}$ for a topological space X is a class of subsets of X such that whenever $C \subset U$, with C compact and U open in X, there is a $P \in \mathcal{P}$ such that $C \subset P \subset U$. E. Michael has recently investigated a class of spaces containing the separable metric spaces which he called $\aleph_0$-spaces and which he defined to be regular, T_1 spaces having countable pseudobases.

In this thesis we generalize the concept of an $\aleph_0$-space in order to include all metric spaces. The principal properties of this extended class of spaces which we call $\aleph_\sigma$-spaces are established. We compare the properties of $\aleph_\sigma$-spaces with those of $\aleph_0$-spaces. In most cases where it is found necessary to weaken a statement concerning $\aleph_\sigma$-spaces, we give examples to show that the corresponding statement for $\aleph_0$-spaces is not true for $\aleph_\sigma$-spaces.

We define a class of subsets $\mathcal{P}$ of a space X to be piecewise σ-locally finite if each $P \in \mathcal{P}$ is a finite union of members of a σ-locally finite class $\mathcal{J}$ of subsets of X. An $\aleph_\sigma$-space is a regular, T_1 space having a piecewise σ-locally finite pseudobase.

The following properties are established for $\aleph_\sigma$-spaces:

A. Each metric space and each $\aleph_0$-space is an $\aleph_\sigma$-space.

B. First countable $\aleph_\sigma$-spaces are metrizable.

C. Locally compact $\aleph_\sigma$-spaces are metrizable.

D. Each open subset of an $\aleph_\sigma$-space is an F_σ .

E. Any subspace of an $\aleph_\sigma$-space is an $\aleph_\sigma$-space.

F. A countable product of $\aleph_\sigma$-spaces is an $\aleph_\sigma$-space.

G. The image of an $\aleph_\sigma$-space, under a perfect mapping, is an $\aleph_\sigma$-space.

H. If X and Y are $\aleph_\sigma$-spaces, then so is any adjunction space $X \cup_f Y$ in which the domain of the attaching map f is compact.

I. If (X,τ) is an $\aleph_\sigma$-space, then so is (X,τ') for any regular, T_1 topology τ' on X yeilding the same compact subsets as τ .

J. If X is a locally compact $\aleph_0$-space and Y is an $\aleph_\sigma$-space, then $\mathcal{C}(X,Y)$ with the compact-open topology is an $\aleph_\sigma$-space.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Professor R. L. McKinney, for his many helpful suggestions concerning the problems of this thesis. His patient editing has saved me from displaying some glaring errors in the first draft of this work.

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CHAPTER 1

INTRODUCTION

In a recent paper [5], E. Michael introduced a class of topological spaces called $\aleph_0$-spaces which contains all separable metric spaces. In this paper, we use the basic notions underlying the concept of $\aleph_0$-spaces to define a class of speces call $\aleph_\sigma$-spaces which contains all metric spaces and all $\aleph_0$-spaces.

1.1 Basic Definitions

Throughout this paper we shall understand that a space is a topological space. The usual notation for a space (X,τ) will be abbreviated to X whenever there is no need to call particular attention to the topology τ.

Definition 1.1.1. *A class $\mathcal{P}$ of subsets of a space X is called a pseudobase for X if, whenever $C \subset U$, with C compact and U open in X, then $C \subset P \subset U$ for some $P \in \mathcal{P}$.*

Definition 1.1.2. *An $\aleph_0$-space is a regular, T_1 space having a countable pseudobase.*

Recall that a base $\mathcal{B}$ for a space X is a class of open

subsets of X such that if $x \in U$ and U is open, then $x \in B \subset U$ for some $B \in \mathcal{B}$. Since a separable metric space has a countable base, it follows that each separable metric space is an $\aleph_0$-space.

A class $\mathfrak{F}$ of subsets of a space X is called locally finite if each point $x \in X$ has a neighborhood which has a non-empty intersection with only finitely many members of $\mathfrak{F}$. Hereafter, when a set A has a non-empty intersection with a set B, we shall say that A meets B. If for each $n \in N = \{1,2,\ldots\}$, $\mathfrak{F}_n$ is a locally finite class, then $\mathfrak{F} = \cup\{\mathfrak{F}_n : n \in N\}$ is said to be σ-locally finite.

<u>Definition 1.1.3</u>. <u>A class $\mathcal{P}$ of subsets of a space X is called piecewise σ-locally finite if each $P \in \mathcal{P}$ is a finite union of members of a σ-locally finite class $\mathfrak{F}$</u>.

<u>Definition 1.1.4</u>. <u>An $\aleph_\sigma$-space is a regular, T_1 space having a piecewise σ-locally finite pseudobase</u>.

In order to see to what extent $\aleph_\sigma$-spaces are like $\aleph_0$-spaces, we summarize the principal properties of $\aleph_0$-spaces here.

A. All separable metric spaces and all their regular, T_1 quotient spaces are $\aleph_0$-spaces.

B. First countable $\aleph_0$-spaces are separable metrizable.

C. Locally compact $\aleph_0$-spaces are separable metrizable.

D. Every $\aleph_0$-space is a regular Lindelöf space (hence paracompact) and every open subset is an F_σ.

E. Each subspace of an $\aleph_0$-space is an $\aleph_0$-space.

F. A countable product of $\aleph_0$-spaces is an $\aleph_0$-space.

G. The image of an $\aleph_0$-space, under a closed, countinuous mapping, is an $\aleph_0$-space.

H. If X and Y are $\aleph_0$-spaces, then so is any adjunction space $X \cup_f Y$.

I. If (X,τ_1) is an $\aleph_0$-space, then so is (X,τ_2) for any regular, T_1 topology τ_2 on X yeilding the same compact subsets as τ_1.

J. If X and Y are $\aleph_0$-spaces, so is the function space $\mathcal{C}(X,Y)$ with the compact-open topology.

K. A regular, T_1 space X is an $\aleph_0$-space iff $k(X)$ is a quotient space of a separable metric space.

L. A regular, T_1 space is an $\aleph_0$-space iff it is a compact-covering image of a separable metric space.

We shall recall the definitions of terms used in the above propositions as the corresponding properties of $\aleph_\sigma$-spaces are discussed. In [5], Michael points out that there are two other classes of spaces having most of the above properties. They are the stratifiable spaces [1]

(which contain all metric spaces, but which are not Lindelöf or separable) and the cosmic spaces (continuous images of separable metric spaces). Neither of these classes satisfy J , even in the special case where X is a closed interval. $\aleph_\sigma$-spaces also share many of the properties possessed by $\aleph_0$-spaces and, when X is an interval, J holds for $\aleph_\sigma$-spaces.

1.2 Notation

The natural numbers $\{1,2,\ldots\}$ will be denoted by N. If A is a subset of a space X, we shall denote the interior of A by A^o, the closure of A by A^-, and the complement of A in X by $X-A$. In cases where the phrase "if and only if" is required, Halmos' "iff" will be used. The usual functional notation will also be employed. That is, a function f whose domain is X and whose range is a subset of Y will be denoted by $f : X \to Y$. If $f(X) = Y$, we shall write $f : X \to\to Y$. If A is a subset of X, then the restriction of f to A (a function whose domain is A and which is equal to $f(a)$ for each $a \in A$) will be denoted by $f|A$.

CHAPTER 2

$\aleph_\sigma$-Spaces Which Are $\aleph_0$ Spaces

Michael has shown [5, Corollary 7.8] that a regular, T_1 space which can be covered by a locally finite, countable collection of closed $\aleph_0$-spaces is an $\aleph_0$-space. Each subspace of an $\aleph_0$-space is also an $\aleph_0$-space [5, E]. We shall use these facts in proving the following proposition which provides the reason for passing over the consideration of spaces having σ-locally finite pseudobases.

<u>Theorem 2.1</u>. <u>A regular, T_1 space X having a σ-locally finite pseudobase $\mathcal{P}$ is a $\aleph_0$-space.</u>

<u>Proof</u>. Let x and y be distinct points of X and let U and V be disjoint open neighborhoods of x and y, respectively. Let $A \subset U$ and $B \subset V$ be closed neighborhoods of x and y, respectively. Finally, let $X_1 = X-A$, $\tilde{X}_1 = X-U$, and define X_2 and $\tilde{X}_2$ in terms of y, similarly.

Since $\mathcal{P}$ is σ-locally finite, x is contained in, at most, countably many members of $\mathcal{P}$. Let $\mathcal{P}(x) = \{P \in \mathcal{P} : x \in P\}$. Now let C be a compact and W open in X_1, with $C \subset W$. Then W is open in X and $C_1 = C \cup \{x\}$ is compact in X. Since $x \in A^o$, there is a $P \in \mathcal{P}(x)$ such that $C_1 \subset P \subset W \cup A^o$. Then $C \subset P \cap X_1 \subset W$. Hence $\mathcal{P}_1 = \{P \cap X_1 : P \in \mathcal{P}(x)\}$ is a countable pseudobase for X_1. Then $\tilde{X}_1$, being a subspace of X_1 is an $\aleph_0$-space. Similarly,

$\tilde{X}_2 = X - V$ is an $\aleph_0$-space. Thus $X = \tilde{X}_1 \cup \tilde{X}_2$ is the union of two closed $\aleph_0$-spaces and that completes the proof.

In the discussions to follow, we shall have frequent need to call attention to the σ-locally finite class of subsets of a space X whose finite unions form a pseudobase for X. We shall say that such a class generates a pseudobase for X or, more briefly, that the class is a generator for X.

Observe that if $\mathfrak{I} = \cup\{\mathfrak{I}_n : n \in N\}$ is a σ-locally finite class of subsets of a space X and if $C \subset X$ is compact, then C meets, at most, countably many members of $\mathfrak{I}$. This follows from the fact that, for each $n \in N$, C can be covered by finitely many neighborhoods of points in C, each of which meets only finitely many members of $\mathfrak{I}_n$. This observation is all that is needed in the following proposition.

Recall that a space is said to be σ-compact if it is a countable union of compact subsets.

<u>Proposition 2.2</u> <u>A σ-compact $\aleph_\sigma$-space is and $\aleph_0$-space.</u>

<u>Proof</u>. If $X = \cup\{X_n : n \in N\}$ is an $\aleph_\sigma$-space and each X_n is compact, then each X_n meets, at most countably many members of a σ-locally finite generator $\mathfrak{I}$ for X. Then $\mathfrak{I}$ itself must be countable so the pseudobase generated by $\mathfrak{I}$ is also countable. That completes the proof.

CHAPTER 3

Closed Generators For $\aleph_\sigma$-Spaces

If C is a compact subset of a regular space X and $C \subset U$, with U open, then there is an open set V such that $C \subset V \subset V^- \subset U$. For C can be covered by finitely many neighborhoods of points in C whose closures are contained in U. Then if X is an $\aleph_\sigma$-space, $\mathfrak{T}$ is a generator for X, and $C \subset U$, with C compact and U open in X, there is a finite union P of members of $\mathfrak{T}$ such that $C \subset P \subset V \subset V^- \subset U$, where V is open. Thus $C \subset P^- \subset U$. Now if $T \in \mathfrak{T}$ is one of the members in the finite class whose union is P, it follows that $T^- \subset P^-$. Observe now that if $\mathfrak{T}_n$ is a locally finite class of subsets of a space X, then $\{T^- : T \in \mathfrak{T}_n\}$ is also locally finite. For if $x \in X$ and U is a neighborhood of x which meets only finitely many $T \in \mathfrak{T}_n$, then $x \in V = X - \cup\{T \in \mathfrak{T}_n : U \cap T = \emptyset\}^-$. Then V is a neighborhood of x which meets T^- for $T \in \mathfrak{T}_n$ only if U meets T. It now follows that a generator for an $\aleph_\sigma$-space may be assumed to consist of closed sets.

A locally finite family $\mathcal{B}$ of subsets of a space X has the closure preserving property. That is, if $\{B_a : a \in A\}$ is any subfamily of $\mathcal{B}$, then $\cup\{B_a^- : a \in A\} = \cup\{B_a : a \in A\}^-$. We need only show that $\cup\{B_a : a \in A\}^- \subset \cup\{B_a^- : a \in A\}$ for the reverse inclusion is true whether or not $\mathcal{B}$ is locally finite. Now a set in X can meet no more members of $\{B_a : a \in A\}$ than it does members of $\mathcal{B}$ so that $\{B_a : a \in A\}$ is

certainly locally finite. Then if $x \in \bigcup\{B_a : a \in A\}^-$ and V is a neighborhood of x which meets only finitely many B_a's, then $x \in \bigcup\{B_a : a \in A \text{ and } V \cap B_a \neq \emptyset\}^- = \bigcup\{B_a^- : a \in A \text{ and } V \cap B_a \neq \emptyset\}$ $\subset \bigcup\{B_a^- : a \in A\}$. Thus $\bigcup\{B_a : a \in A\}^- \subset \bigcup\{B_a^- : a \in A\}$.

Recall that an F_σ subset of a space is a countable union of closed subsets. The following proposition (which shows that $\aleph_\sigma$-spaces share a part of property D of $\aleph_0$-spaces) is an easy consequence of the definition of $\aleph_\sigma$-spaces and the above remarks.

<u>Proposition 3.1</u> Every open subset of an $\aleph_\sigma$-space is an F_σ.

<u>Proof.</u> Let U be an open subset of an $\aleph_\sigma$-space X with a σ-locally finite closed generator $\mathcal{J} = \bigcup\{\mathcal{J}_n : n \in N\}$. For each $n \in N$, let $\mathcal{J}_n(U) = \{T \in \mathcal{J}_n : T \subset U\}$ and let $F_n = \bigcup\{T : T \in \mathcal{J}_n(U)\}$. Then each F_n is closed in view of the preceding remarks an $U = \bigcup\{F_n : n \in N\}$ (since each $\{x\} \subset U$ is compact).

CHAPTER 4

Paracompact and Metrizable $\aleph_\sigma$-Spaces

4.1 Paracompact $\aleph_\sigma$-Spaces

If $\mathcal{U}$ is an open cover of a space X, then a cover $\mathcal{V}$ of X is said to refine $\mathcal{U}$ or to be a refinement of $\mathcal{U}$ if each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. A Hausdorff space X is called paracompact iff each open cover of X has an open locally finite refinement. There are several equivalents to paracompactness for regular spaces, one of which is that a regular space X is paracompact iff each open cover of X has a locally finite refinement [4, Lemma 1, b].

Unfortunately, not all $\aleph_\sigma$-spaces are paracompact as we show in Example 4.3.1. However, there is a special class of $\aleph_\sigma$-spaces, each of which is paracompact. Moreover, just as for $\aleph_0$-spaces, the first countable and the locally compact $\aleph_\sigma$-spaces are metrizable.

Let us begin by recalling a few definitions. If x is a point in a space X, a base at x is a system of neighborhoods of x $\{U_a : a \in A\}$ such that if $x \in U$ and U is open, then $x \in U_a \in U$ for some $a \in A$. A first countable space X is one for which there is a countable base at x for each $x \in X$. When considering first countable spaces, we shall always assume that if $\{U_n(x) : n \in N\}$ is a base at x, then $U_{n+1}(x) \subset U_n(x)$ for each $n \in N$. This can be arranged by letting

$V_m(x) = \bigcap_{n=1}^{m} U_n(x)$ for each $m \in N$ and taking $\{V_m(x) : m \in N\}$ for a base at x.

A locally compact space is one in which each point has a compact neighborhood. Each compact space is then locally compact.

Finally, an r-space X is one in which each point $x \in X$ has a countable system of neighborhoods $\{U_n(x) : n \in N\}$ such that if $x_n \in U_n(x)$ for each $n \in N$, then $\{x_1, x_2, x_3, \ldots\}$ has compact closure. It is easy to see that each first countable, T_1 space and each locally compact Hausdorff space is an r-space.

In view of the metrization theorem of J. Nagata [6, Theorem 1] and Yu. Smirnov [7, Theorem 1] which states that a regular, T_1 space is metrizable iff it has a σ-locally finite base, the definition of $\aleph_\sigma$-spaces is tailored to fit metric spaces. However, we shall include a proof of this and another equally obvious assertion for the sake of completeness before carrying on with the principal results of this chapter.

<u>Proposition 4.1.1</u>. <u>Each metric space and each $\aleph_0$-space is an $\aleph_\sigma$-space</u>.

<u>Proof</u>. If X is a metric space, then by the aforementioned result of Nagata and Smirnov, X has a σ-locally finite base $\mathcal{B}$. If $\mathcal{P}$ is the class of finite unions of members of $\mathcal{B}$, then clearly, $\mathcal{P}$ is a pseudobase for X and hence X is an $\aleph_\sigma$-space.

If X is an $\aleph_0$-space, then since any countable class of subsets of X is σ-locally finite, X is also an $\aleph_\sigma$-space.

In order to establish the principal results of this chapter, we shall need the following lemmas.

Lemma 4.1.2. *Let X be an $\aleph_\sigma$-space which is an r-space and let $\mathcal{J} = \bigcup\{\mathcal{J}_n : n \in N\}$ be a σ-locally finite generator for X. For each $m \in N$, let $\mathcal{J}'_m = \bigcup\{\mathcal{J}_n : n \leq m\}$. Then if U is open in X and $x \in U$, there is an $m \in N$ such that $x \in [F(m,U)]^o$, where $F(m,U) = \bigcup\{T \in \mathcal{J}'_m : T \subset U\}$.*

Proof. Suppose not. Then there is an $x \in X$ and an open neighborhood U of x such that $x \in [F(m,U)]^o$ is false for every $m \in N$. Let $\{U_n(x) : n \in N\}$ be a sequence of neighborhoods of x such that if $x_n \in U_n(x)$ for each $n \in N$, then $\{x_1, x_2, \ldots,\}^-$ is compact. Let V be a neighborhood of x such that $V^- \subset U$ and for each $n \in N$, let $V_n = V \cap U_n(x)$. Since $V_n \not\subset F(n,U)$, we can choose $x_n \in V_n - F(n,U)$. Then $C = \{x_1, x_2, \ldots\}^-$ is compact and $C \subset V^- \subset U$. Then there is a finite union $\bigcup_{k=1}^{r} T_k$ of members of $\mathcal{J}$ such that $C \subset \bigcup_{k=1}^{r} T_k \subset U$. But if $m = \max.\{\min. n : T_k \in \mathcal{J}_n ; k = 1,2,\ldots,r\}$, then $C \subset F(m,U)$ which contradicts the choice of $x_m \in C$ and that completes the proof.

Lemma 4.1.3. *Let X be a first countable $\aleph_\sigma$-space with σ-locally finite closed generator $\mathcal{J} = \bigcup\{\mathcal{J}_n : n \in N\}$. For each $m \in N$, let*

$\mathcal{T}'_m = \bigcup\{\mathcal{T}_n : n \leq m\}$. Then if U is an open neighborhood of a point $x \in X$, there is an $m \in N$ such that $x \in [St(x, \mathcal{T}'_m, U)]^o$, where $ST(x, \mathcal{T}'_m, U) = \bigcup\{T \in \mathcal{T}'_m : x \in T \subset U\}$.

Proof. Suppose there is an $x \in X$ and an open neighborhood U of x such that $x \in [St(x, \mathcal{T}'_m, U)]^o$ is false for each $m \in N$. Let $\{U_n(x) : n \in N\}$ be all those members of a countable base at x such that each $U_n(x) \subset U$. Choose $x_n \in U_n(x) - St(x, \mathcal{T}'_n, U)$. Then $x_n \to x$ and $C = \{x, x_1, x_2, \ldots,\}$ is compact. Since $C \subset U$ there is a finite union $P = \bigcup\{T_a : a \in A\}$ of members of $\mathcal{T}$ such that $C \subset P \subset U$. Let $B \subset A$ be the set of indices such that $x \in T_b$ for each $b \in B$. Then if $n_o \in N$, there is an $n > n_o$ such that $x_n \in P' = \bigcup\{T_b : b \in B\}$. For if not, then $\{x_n : n > n_o\}$ is in $P'' = \bigcup\{T_a : a \in A-B\}$ which is closed and does not contain x . But this contradicts $x_n \to x$ since $X-P''$ is a neighborhood of x .

Now let $m = \max\limits_{b \in B} \{\min n : T_b \in \mathcal{T}_n\}$. Then $P' \subset St(x, \mathcal{T}'_m, U)$. Note that if $k \geq j$, then $St(x, \mathcal{T}'_j, U) \subset St(x, \mathcal{T}'_k, U)$. Then there is a $k > m$ such that $x_k \in P' \subset St(x, \mathcal{T}'_m, U) \subset St(x, \mathcal{T}'_k, U)$ which contradicts the choice of x_k and that completes the proof.

Theorem 4.1.4 . If X is a $\aleph_\sigma$-space which is an r-space, then X is paracompact.

Proof. We shall show that every open cover of X has a locally finite refinement.

Let $\mathcal{U}$ be an open cover of X and for each $n \in N$, let $\mathcal{T}_n(\mathcal{U}) = \{T \in \mathcal{T}_n : T \subset U$ for some $U \in \mathcal{U}\}$ where $\mathcal{T} = \cup\{\mathcal{T}_n : n \in N\}$ is a σ-locally finite generator for X. For each $m \in N$ and for each $T \in \mathcal{T}_m(\mathcal{U})$, let $\hat{T} = T - \cup\{T' : T' \in \mathcal{T}_n(\mathcal{U})\ ;\ n < m\}$. Let $\hat{\mathcal{T}} = \{\hat{T} : T \in \mathcal{T}_n(\mathcal{U})$ for some $n \in N\}$. Then $\hat{\mathcal{T}}$ is a locally finite refinement of $\mathcal{U}$.

First we show that $\hat{\mathcal{T}}$ is locally finite. Let $x \in X$ and let $U \in \mathcal{U}$ be a neighborhood of x. By lemma 4.1.2, there is an $m \in N$ such that $x \in [F(m,U)]^{\circ}$, where $F(m,U) = \cup\{T \in \bigcup_{n=1}^{m} \mathcal{T}_n : T \subset U\}$. Then $[F(m,U)]^{\circ} \cap \hat{T} = \emptyset$ if $T \in \mathcal{T}_k(\mathcal{U})$ for $k > m$. If for each $n \leq m$, $V_n(x)$ is a neighborhood of x which meets only finitely many $T \in \mathcal{T}_n$, then $[F(m,u)]^{\circ} \cap \bigcap_{n=1}^{m} V_n(x)$ is a neighborhood of x which meets only finitely many $\hat{T} \in \hat{\mathcal{T}}$.

Now if $x \in X$, there is a first n such that $x \in T \in \mathcal{T}_n(\mathcal{U})$. Then $x \in \hat{T}$ for this T. Therefore, $\hat{\mathcal{T}}$ refines $\mathcal{U}$ since, clearly each $\hat{T} \in \hat{\mathcal{T}}$ is a subset of some $U \in \mathcal{U}$ and that completes the proof.

4.2 Metrizable $\aleph_\sigma$-Spaces

A space X is said to be locally metrizable if each point $x \in X$ has a neighborhood which is a metrizable subspace of X. Smirnov [8, Theorem 3] showed that a locally metrizable Hausdorff space is metrizable iff it is paracompact. If we knew that the property of being an $\aleph_\sigma$-space was hereditary, there would be an easy corollary to this and proposition 4.1.4,

in view of the above theorem of Smirnov.

<u>Proposition 4.2.1</u>. <u>Each subspace of an $\aleph_\sigma$-space is and $\aleph_\sigma$-space.</u>

<u>Proof</u>. Let X be an $\aleph_\sigma$-space with σ-locally finite generator $\mathfrak{I}$ and let $A \subset X$. Then $\mathfrak{I}(A) = \{T \cap A : T \in \mathfrak{I}\}$ is a σ-locally finite generator for A with the relative topology.

Clearly, $\mathfrak{I}(A)$ is σ-locally finite in A and if $C \subset A$ is compact, then C is compact in X also. Thus if $C \subset U \cap A$, with U open in X, there is a finite union $\bigcup_{n=1}^{k} T_n$ of members of $\mathfrak{I}$ such that $C \subset \bigcup_{n=1}^{k} T_n \subset U$. Then $C \subset \bigcup_{n=1}^{k} (T_n \cap A) \subset U \cap A$ and that completes the the proof.

<u>Corollary 4.2.2</u>. <u>A locally compact $\aleph_\sigma$-space is metrizable.</u>

<u>Proof</u>. Let X be a locally compact $\aleph_\sigma$-space. Let $x \in X$ and let C be a compact neighborhood of x. Then C is a compact $\aleph_\sigma$-space by 4.2.1 and by 2.2, C is an $\aleph_0$-space. By [5, C], C is a separable metrizable subspace of X. Thus X is locally metrizable. Now 4.1.4 shows that X is paracompact (since each locally compact Hausdorff space is an r-space) and hence, X is metrizable by the aforementioned theorem of Smirnov.

<u>Theorem 4.2.3</u>. <u>An $\aleph_\sigma$-space</u> X <u>is metrizable iff it is first countable.</u>

Proof. The necessity of the condition is immediate. By the Nagata-Smirnov metrization theorem, it is sufficient to show that a first countable $\aleph_\sigma$-space has a σ-locally finite base.

Let $\mathcal{J} = \cup\{\mathcal{J}_n : n \in N\}$ be a σ-locally finite closed generator for X. For each $m \in N$, let $\mathcal{J}'_m = \cup\{\mathcal{J}_n : n \leq m\}$ and for each $x \in X$ and each $n \in N$, let $W_n(x) = X - \cup\{T \in \mathcal{J}'_n : x \notin T\}$. Then $W_n(x)$ is an open neighborhood of x for each $x \in X$ and each $n \in N$ by the closure preserving property of $\mathcal{J}'_n$ (which is clearly a locally finite class, being a finite union of locally finite classes). Let $\{\tilde{U}_n(x) : n \in N\}$ be a countable open base at x for each $x \in X$. Finally, for each $x \in X$ and each $n \in N$, let $U_n(x) = W_n(x) \cap \tilde{U}_n(x)$. Then $\{U_n(x) : n \in N\}$ is a countable open base at x for each $x \in X$ and $U_n(x)$ meets only those members of $\mathcal{J}'_n$ which contain x in view of the definition of $W_n(x)$.

By 4.1.4, X is paracompact. Then for each $n \in N$, let $\mathcal{V}_n$ be a locally finite open refinement of $\mathcal{U}_n = \{U_n(x) : x \in X\}$. Now for each pair $(n,m) \in N \times N$, each $V \in \mathcal{V}_n$, and each $x \in V$, let $\tilde{V}(x,n,m) = [St(x,\mathcal{J}'_n,U_m(x))]^\circ \cap V$. Since each $V \in \mathcal{V}_n$ is contained in some $U_n(x)$, each $V \in \mathcal{V}_n$ meets only finitely many $T \in \mathcal{J}'_n$. Then for n and m fixed, there are only finitely many $\tilde{V}(x,n,m)$'s for each $V \in \mathcal{V}_n$ since if $x \in V$, then V meets every $T \in \mathcal{J}'_n$ containing x. Thus $\mathcal{V}_{n,m} = \{\tilde{V}(x,n,m) : V \in \mathcal{V}_n\}$ is a locally finite open class for each pair $(n,m) \in N \times N$. We shall now show that $\mathcal{V} = \cup\{\mathcal{V}_{n,m} : (n,m) \in N \times N\}$ is a base for X.

Let U be an open set containing x . Let $m \in N$ be such that $U_m(x) \subset U$. By Lemma 4.1.3, let $n \in N$ be such that $x \in [St(x, \mathcal{J}'_n, U_m(x))]^o$. Since $\mathcal{V}_n$ refines $\mathcal{U}_n$, $x \in V$ for some $V \in \mathcal{V}_n$ and hence $x \in \tilde{V}(x,n,m) \in \mathcal{V}_{n,m}$ for this V . Since $St(x, \mathcal{J}'_n, U_m(x)) \subset U_m(x)$, $\tilde{V}(x,n,m) \subset U$ and that completes the proof.

4.3 Examples

<u>Example 4.3.1</u>. <u>An $\aleph_\sigma$-space which is not normal (and therefore, not paracompact)</u>.

For each $n \in N$, let $X_n = \{(x,y) : x \in R, y \in R,$ and $n-1 < y < n\}$, where R denotes the real numbers. Let $X_o = \{(x,0) : x \in R\}$. Let $X = \cup\{X_n : n \geq 0\}$ and let each X_n $(n > 0)$ have the usual topology. Define neighborhoods of points $(x,0) \in X_o$ as follows: For each $n > 0$, let $C_n(x)$ be an open disk in X_n , centered at (x,y) for some $y \in (n-1,n)$, and tangent to $y = n-1$. (That is, the closure of $C_n(x)$ in the usual upper half-plane is to be tangent to $y = n-1$.) Let a basis for neighorhoods of $(x,0)$ be those sets which are unions of all but finitely many $C_n(x)$'s and $\{(x,0)\}$.

It is easy to verify that X with the topology so defined is a regular, T_1 space. If $\mathcal{P}_n$ is a countable pseudobase for X_n , $n > 0$, (each such X_n is a separable metric space and therefore, an $\aleph_o$-space) and if $\mathcal{J}_o = \{\{(x,0)\} : x \in R\}$, then $\mathcal{J} = \cup\{\mathcal{P}_n : n \in N\} \cup \mathcal{J}_o$ is a σ-locally finite generator for X . $\mathcal{J}$ is σ-locally finite

because each point $p \in X$ has a neighborhood which meets, at most, one member of $\mathcal{J}_o$ and each $\mathcal{P}_n$ is countable. Now observe that for $n > 0$, X_n is both open and closed in X since each point $p \in X_o$ has a neighborhood which does not meet X_n. Furthermore, a compact $C \subset X$ can contain, at most, finitely many points $p \in X_o$. (The restriction of the topology of X to X_o yeilds the discrete topology on X_o.) Then if C is compact in X, $C \cap X_n = \emptyset$ for all but finitely many n. The assertion that $\mathcal{J}$ is a generator for X now follows immediately.

We shall now show that X is not normal by appealling to the following result of F. B. Jones [2, p. 144, Example 3]: If a space X contains a dense set D and a closed discrete subspace S such that $|S| \geq 2^{|D|}$, then X is not normal. (Here we use $|A|$ to denote the cardinal number of a set A.)

Since X_o is a closed discrete subspace of X in our example and since $D = \{(x,y) \in X : x$ and y rational$\}$ is dense in X, it follows from Jones' result that X is not normal.

<u>Example 4.3.2. A first countable, regular, Lindelöf space which is not an $\aleph_\sigma$-space.</u>

Let X be the upper half-plane and let $A \subset X$ be the x-axis. Let ρ be the usual metric on X and let neighborhoods of points in X-A be the usual ones. A base at a point $p \in A$ consists of all sets $\{N(p, \frac{1}{n}) : n \in N\}$, where $N(p, \frac{1}{n})$ consists of p together with all

points q such that $\rho(p,q) < \frac{1}{n}$ and such that q lies beneath the union of the two rays in X which emanate from p and have slopes $\frac{1}{n}$ and $-\frac{1}{n}$, respectively. The space X is called the "butterfly space".

It is easy to check that X is a regular, Hausdorff, Lindelöf space which is first countable. $X-A$ and A retain their usual topologies and hence are separable metrizable. Since X is separable but does not have a countable base (any base must contain neighborhoods of each $p \in A$), X is not metrizable. Then by 4.2.3, X is not an $\aleph_\sigma$-space.

CHAPTER 5

Products of $\aleph_\sigma$-Spaces

Let $\mathcal{U}$ be a class of subsets of a space X. $\mathcal{P}$ is a $\mathcal{U}$-pseudobase for X if, whenever $C \subset U$, with C compact and $U \in \mathcal{U}$, there is a $P \in \mathcal{P}$ such that $C \subset P \subset U$. A $\mathcal{U}$-generator for X is a class of subsets of X whose finite unions form a $\mathcal{U}$-pseudobase for X.

Lemma 5.1. Let X be a Hausdorff space and let $\mathcal{S}$ be a sub-base for X. Then X has a σ-locally finite generator iff it has a σ-locally finite $\mathcal{S}$-generator.

Proof. If $\mathcal{T}$ is a generator for X, then $\mathcal{T}$ is also an $\mathcal{S}$-generator for X, so that the necessity of the condition is immediate.

Now suppose that $\mathcal{T} = \bigcup\{\mathcal{T}_n : n \in N\}$ is a σ-locally finite $\mathcal{S}$-generator for X. Let $\hat{\mathcal{T}}$ be the class of all finite intersections of members of $\mathcal{T}$. Let M be the class of all finite subsets of N. For each $s \in M$, let $\hat{\mathcal{T}}_s$ be the class of all finite intersections of members of $\bigcup\{\mathcal{T}_n : n \in s\}$. Then if $x \in X$ and $V_n(x)$ is a neighborhood of x which meets only finitely many $T \in \mathcal{T}_n$, $W(x) = \bigcap\{V_n(x) : n \in s\}$ is a neighborhood of x which meets only finitely many $T \in \hat{\mathcal{T}}_s$. Thus $\hat{\mathcal{T}} = \bigcup\{\hat{\mathcal{T}}_s : s \in M\}$ is σ-locally finite. We shall show that $\hat{\mathcal{T}}$ is a generator for X.

First suppose that $C \subset U \in \mathcal{B}$, where C is compact and $\mathcal{B}$ is a base for X consisting of all finite intersections of members of $\mathcal{S}$. Then $U = \bigcap_{i=1}^{n} \{S_i : S_i \in \mathcal{S}\}$ and for each $i = 1,2,\ldots,n$; there is a finite union $P_i = \bigcup_{j=1}^{r_i} T_{\alpha_j^i}$ of members of $\mathcal{J}$ such that $C \subset P_i \subset S_i$. Then $C \subset \bigcap_{i=1}^{n} P_i \subset U$. If $R_i = \{\alpha_1^i,\ldots,\alpha_{r_i}^i\}$ for each $i = 1,2,\ldots,n$; and $A = R_1 \times R_2 \times \ldots \times R_n$, then $\bigcap_{i=1}^{n} \bigcup_{j=1}^{r_i} T_{\alpha_j^i} = \bigcup \left\{ \left(\bigcap_{i=1}^{n} T_{\alpha_j^i} \right)_a : a \in A \right\}$. Then since A is finite, $\hat{\mathcal{J}}$ forms a $\mathcal{B}$-generator for X.

If U is an arbitrary open set, let $\{B_i : i = 1,2,\ldots,n\}$ be a finite subclass of $\mathcal{B}$ such that $C \subset \bigcup_{i=1}^{n} B_i \subset U$. Since C is normal, there are closed subsets $C_i \subset B_i$, $i = 1,2,\ldots,n$; such that $C = \bigcup_{i=1}^{n} C_i$ [9, p. 152, 6.1]. According to the previous paragraph, there is a finite union K_i of members of $\hat{\mathcal{J}}$ such that $C_i \subset K_i \subset B_i$; $i = 1,2,\ldots,n$. Then $K = \bigcup_{i=1}^{n} K_i$ is a finite union of members of $\hat{\mathcal{J}}$ and $C \subset K \subset U$, which completes the proof.

The following result on countable products shows that the situation is the same as for $\aleph_0$-spaces.

<u>Theorem 5.2.</u> <u>A countable product of $\aleph_\sigma$-spaces is an $\aleph_\sigma$-space.</u>

<u>Proof.</u> Let $\{X_n : n \in N\}$ be a sequence of $\aleph_\sigma$-spaces and for each $n \in N$, let $\mathcal{J}_n = \bigcup\{\mathcal{J}_{n,m} : m \in N\}$ be a σ-locally finite generator for X_n. Let $\mathcal{R}_{n,m} = \{\pi_n^{-1}(T) : T \in \mathcal{J}_{n,m}\}$ for each pair $(n,m) \in N \times N$, where π_n is the projection from $X = \prod_{n=1}^{\infty} X_n$ onto the n^{th} coordinate

space X_n . Then $\mathcal{R} = \cup\{\mathcal{R}_{n,m} : (n,m) \in N \times N\}$ is a σ-locally finite $\mathcal{S}$-generator for X , where $\mathcal{S}$ is the sub-base for X consisting of all sets $\pi_n^{-1}(U_n)$, with U_n open in X_n and $n \in N$.

It is easy to see that $\mathcal{R}_{n,m}$ is locally finite. For if x_n is the n^{th} coordinate of $x \in X$ and $U(x_n)$ is a neighborhood of x_n which meets only finitely many $T \in \mathcal{J}_{n,m}$, then $\pi_n^{-1}(U(x_n))$ is a neighborhood of x which meets only finitely many members of $\mathcal{R}_{n,m}$.

Now if $C \subset \pi_n^{-1}(U_n)$, with C compact in X and U_n open in X_n , then $\pi_n(C) \subset U_n$ and $\pi_n(C)$ is compact in X_n . Thus there is a finite union $\bigcup_{k=1}^{r} T_k$ of members of $\mathcal{J}_n$ such that $\pi_n(C) \subset \bigcup_{k=1}^{r} T_k \subset U_n$. Then $C \subset \pi_n^{-1}\left(\bigcup_{k=1}^{r} T_k\right) = \bigcup_{k=1}^{r} \pi_n^{-1}(T_k) \subset \pi_n^{-1}(U_n)$ so that $\mathcal{R}$ is indeed an $\mathcal{S}$-generator for X . Now a product of regular, T_1 spaces is a regular, T_1 space and hence, by Lemma 5.1, X is an $\aleph_\sigma$-space.

CHAPTER 6

Mappings of $\aleph_\sigma$-Spaces

6.1 Perfect Mappings

By a mapping $f : X \to Y$, we mean that f is a continuous function from X into Y . A mapping $f : X \to\to Y$ is called compact if $f^{-1}(y)$ is compact for each $y \in Y$ and f is said to be perfect if it is compact and closed (maps closed subsets of X onto closed subsets of Y).

If R is an equivalence relation in $X \times X$ and $R[x]$ is the equivalence class of x for each $x \in X$, we call $\mathcal{D} = \{R[x] : x \in X\}$ the decomposition of X induced by R . Clearly, if $f : X \to Y$ is a function, then $R(f) = \{(x,y) : f(x) = f(y)\}$ is an equivalence relation in $X \times X$ and the decomposition of X induced by $R(f)$ is $\{f^{-1}(y) : y \in f(X)\}$.

A decomposition $\mathcal{D}$ of a space X is said to be upper semi-continuous if, whenever a member $d \in \mathcal{D}$ is continued in an open set U , there is an open set $V \subset U$ such that $d \subset V$ and V is the union of members of $\mathcal{D}$.

<u>Theorem 6.1.1</u>. <u>The image, under a perfect mapping, of an $\aleph_\sigma$-space is an $\aleph_\sigma$-space.</u>

Proof. Let X be an $\aleph_\sigma$-space with σ-locally finite generator $\mathfrak{T} = \cup\{\mathfrak{T}_n : n \in N\}$ and let $f : X \to\to Y$ be a perfect mapping.

First, Y is T_1 since f is closed and Y is regular since f is perfect [2, p. 235, 5.2 (2)] .

We shall now show that $\mathfrak{F}_n = \{f(T) : T \in \mathfrak{T}_n\}$ is locally finite in Y for each $n \in N$ and that $\mathfrak{F} = \cup\{\mathfrak{F}_n : n \in N\}$ is a generator for Y .

Let $y \in Y$ and let V be a neighborhood of $f^{-1}(y)$ which meets only finitely many $T \in \mathfrak{T}_n$. (There is such a V since $f^{-1}(y)$ is compact.) Since $\mathfrak{D} = \{f^{-1}(y) : y \in Y\}$ is upper semi-continuous [10, Theorem 5], there is an open W which is the union of members of $\mathfrak{D}$ such that $f^{-1}(y) \subset W \subset V$. $f(W)$ is an open neighborhood of y since $W = f^{-1}f(W)$ and Y has the quotient topology induced by f [3, p. 95, Theorem 8]. Suppose that $f(W)$ meets $f(T)$ for some $T \in \mathfrak{T}_n$. Then $W \cap f^{-1}f(T) \neq \emptyset$. If $x \in W \cap f^{-1}f(T)$, then there is an $x' \in T$ such that $f(x) = f(x')$. But in view of the definition of W , if $x \in W$, then $\{x' : f(x) = f(x')\} \subset W$. Hence W (and therefore, V) must meet T . Then $f(W)$ meets $f(T) \in \mathfrak{F}_n$ only if V meets $T \in \mathfrak{T}_n$. Thus $\mathfrak{F}_n$ is locally finite in Y for each $n \in N$.

Now if $C \subset Y$ is compact, so also is $f^{-1}(C)$ in X [2, p. 256, 5.3 (2)] . Then if $C \subset U$, with C compact and U open in Y , $f^{-1}(C) \subset f^{-1}(U)$ and therefore, there is a finite union P of members of $\mathfrak{T}$ such that $f^{-1}(C) \subset P \subset f^{-1}(U)$. Thus $C \subset f(P) \subset U$ and clearly,

$f(P)$ is a finite union of members of $\mathfrak{F}$.

If A is a subset of a space X, then a point $x \in X$ is called a boundary point of A if every neighborhood of x meets both A and $X-A$. The boundary of A, denoted by ∂A, is the set of all boundary points of A.

A mapping f is said to be peripherally compact or simply, π-compact if $\partial f^{-1}(y)$ is compact for each y in the range of f. A.H. Stone has shown [9] that if X is a metric space and $f : X \to\to Y$ is a closed mapping, then Y is metrizable iff f is π-compact. Hence the requirement in 6.1.1 that f be compact may be weakened if X is a metric space. Example 6.3.1 shows that, unlike $\aleph_0$-spaces, an arbitrary closed mapping of an $\aleph_\sigma$-space may fail to have an $\aleph_\sigma$-space for its range.

6.2 Adjunction Spaces

If $\{X_a : a \in A\}$ is a family of spaces and for each $a \in A$, $X'_a = \{a\} \times X_a$, then the family $\{X'_a : a \in A\}$ is pairwise disjoint. Clearly, X_a and X'_a are homeomorphic for each $a \in A$. The disjoint (free) union X of the family $\{X_a : a \in A\}$ is the set $\cup\{X'_a : a \in A\}$ with the weak topology determined by the spaces X'_a; that is, U is open in X iff $U \cap X'_a$ is open in X'_a for each $a \in A$. X is written as $\Sigma\{X'_a : a \in A\}$.

Now let X and Y be disjoint spaces, A a closed subset of

X , and $f : A \to Y$ continuous. The adjunction space $X \cup_f Y$ is the quotient space of $X + Y$ obtained by identifying each $a \in A$ with $f(a)$.

We now proceed to prove for $\aleph_\sigma$-spaces a version of property H of $\aleph_0$-spaces. (If X and Y are $\aleph_0$-spaces, then so is any adjunction space $X \cup_f Y$.) As was the case for closed mappings of $\aleph_\sigma$-spaces, the form of the statement must be weakened as Example 6.3.1 shows [2, p. 128, Example 1]. We shall need the following lemmas.

<u>Lemma 6.2.1</u>. <u>If</u> $\{X_a : a \in A\}$ <u>is a family of</u> $\aleph_\sigma$<u>-spaces, then so is</u> $X = \Sigma\{X_a' : a \in A\}$.

<u>Proof</u>. For each $a \in A$, let $\mathfrak{J}_a = \cup\{\mathfrak{J}_{n,a} : n \in N\}$ be a σ-locally finite generator for $X_a' = \{a\} \times X_a$. If for each $n \in N$, $\mathfrak{J}_n = \cup\{\mathfrak{J}_{n,a} : a \in A\}$, then clearly, $\mathfrak{J}_n$ is locally finite in X . Thus $\mathfrak{J} = \cup\{\mathfrak{J}_n : n \in N\}$ is a σ-locally finite generator for X . (Note that a compact subset of X can have a non-empty intersection with, at most, finitely many X_a''s .) The regular, T_1 character of X is also immediate.

Now let $X \cup_f Y$ be an adjunction space and let $p : X + Y \to\to X \cup_f Y$ be the projection. It is easy to verify that for $C \subset X + Y$, $p^{-1}p(C) = C \cup f(C \cap A) \cup f^{-1}f(C \cap A) \cup f^{-1}(C \cap Y)$, where $f : A \to Y$ is continuous and $A \subset X$ is closed. Then $p^{-1}p(C) \cap Y = (C \cap Y) \cup f(C \cap A)$ and $p^{-1}p(C) \cap X = (C \cap X) \cup f^{-1}(p^{-1}p(C) \cap Y)$. Now if $C \cap X$ is closed, then $p^{-1}p(C)$ is closed in $X + Y$ iff $(C \cap Y) \cup f(C \cap A)$ is closed in Y . We have then, the following lemma which can be found in [2, p. 128, 6.2].

Lemma 6.2.2. Let $X \cup_f Y$ be an adjunction space and let $p : X + Y \to\to X \cup_f Y$ be the projection. If $C \subset X + Y$ is such that $C \cap X$ is closed, then $p(C)$ is closed iff $(C \cap Y) \cup f(C \cap A)$ is closed.

Now if $A \subset X$ is compact and Y is Hausdorff, then for a closed $F \subset X + Y$, $p(F)$ is closed since $F \cap Y$ and $f(F \cap A)$ are both closed. Furthermore, for each $z \in X \cup_f Y$, $p^{-1}(z)$ is compact since $f^{-1}f(a)$ is compact (closed in A) for each $a \in A$ and both $p|Y$ and $p|X-A$ are homeomorphisms [2, p. 128, 6.3]. Thus if X and Y are $\aleph_\sigma$-spaces, $A \subset X$ is compact, and $f : A \to Y$ is continuous, then $X \cup_f Y$ is an $\aleph_\sigma$-space since the projection $p : X + Y \to\to X \cup_f Y$ is a perfect mapping. We have then, the following weaker form of H.

Corollary 6.2.3. If X and Y are $\aleph_\sigma$-spaces, then so is any adjunction space $X \cup_f Y$ in which the domain of the attaching map f is compact.

Corollary 6.2.4. If Y is a space covered by a locally finite closed family $\{X_a : a \in A\}$ of $\aleph_\sigma$-spaces, then Y is an $\aleph_\sigma$-space.

Proof. Let $X = \Sigma\{X_a' : a \in A\}$. Then X is an $\aleph_\sigma$-space by Lemma 6.2.1. We claim that the projection $p : X \to\to Y$ is a perfect mapping. First, p is closed since for any $F \subset X$, $p(F) = \cup\{p(F \cap X_a') : a \in A\}$ where $X_a' = \{a\} \times X_a$. If F_a' is the homeomorphic image of $F \cap X_a'$ in X_a for each $a \in A$, then $p(F \cap X_a') = F_a'$. Now if F is closed in X, F_a' is closed in X_a for each $a \in A$ and thus $\cup\{F_a' : a \in A\} = p(F)$ is closed in Y since $\{X_a : a \in A\}$ is a closed locally finite family.

Now the locally finite character of $\{X_a : a \in A\}$ also insures that for each $y \in Y$, $p^{-1}(y)$ is finite and therefore, compact. The corollary now follows from Theorem 6.1.1 .

6.3 Examples

Example 6.3.1. Let I be the closed unit interval and let A be an uncountable set with the discrete topology. Then $X = A \times I$ is a metrizable space and hence, is an $\aleph_\sigma$-space. If Y is the quotient space of X obtained by identifying the points of the set $\{(a,0) : a \in A\}$, then Y is not an $\aleph_\sigma$-space but the projection $p : X \twoheadrightarrow Y$ is closed.

Proof. In order to see that Y is not an $\aleph_\sigma$-space, let us proceed as follows: Say that a point x in a space X is accessible by a path P in X (a path in X is continuous image of I^o) if there is a sequence $s = \{x_n : n \in N\}$ in $P - \{x\}$ such that $s \to x$. Then if X is a T_1 space containing a point x which is accessible by an uncountable, pair-wise disjoint family of paths $\{P_a : a \in A\}$ such that $x_a \in P_a - \{x\}$ implies that $\{x_a : a \in A\}$ is locally finite, then X is not an $\aleph_\sigma$-space.

Suppose that X is an $\aleph_\sigma$-space with σ-locally finite closed generator $\mathfrak{I}$. Let $\mathfrak{I}(x) = \{T \in \mathfrak{I} : x \in T\}$. Then $\mathfrak{I}(x)$ is countable. Choose a relation $<$ which well-orders A. For each $a \in A$, let $s(a) = \{x_n(a) : n \in N\}$ be a sequence in $P_a - \{x\}$ such that $s(a) \to x$. Let a_1 be the $<$ first member of A. As in the proof

of Lemma 4.3.1, there is a $T_{a_1} \in \mathcal{J}(x)$ and a point $x(a_1) \in s(a_1) \cap T_{a_1}$. Then $X - \{x(a_1)\}$ is open. Let $a_2 \in A$ be the $<$ first element of A such that $a_2 > a_1$. Again as in the proof of 4.3.1, we can pick $T_{a_2} \in \mathcal{J}(x)$ and $x(a_2) \in s(a_2) \cap T_{a_2}$, where $T_{a_2} \subset X - \{x(a_1)\}$. For $a \in A$, suppose that $x(b)$ has been chosen in $s(b) \cap T_b$ and $T_b \subset X - \bigcup_{c<b} \{x(c)\}$, where $T_b \in \mathcal{J}(x)$ for each $b < a$. Then, by the proof of 4.3.1, there is some $T_a \in \mathcal{J}(x)$ such that $T_a \subset X - \bigcup_{b<a} \{x(b)\}$ and $s(a) \cap T_a \neq \emptyset$. Evidently, the T_a's thus chosen are all distinct, which contradicts the countability of $\mathcal{J}(x)$. Thus X is not an $\aleph_\sigma$-space.

Now we see easily that the space Y of our example is a T_1 space containing a "point" $\{(a,0) : a \in A\}$ which is accessible by $I_a = \{a\} \times (I - \{0,1\})$ for each $a \in A$ and for which $x_a \in I_a$ implies that $\{x_a : a \in A\}$ is locally finite.

The preceding example also shows that a regular, T_1 quotient space of a metric space need not be an $\aleph_\sigma$-space.

<u>Example 6.3.2</u>. <u>A paracompact $\aleph_\sigma$-space which is neither an $\aleph_0$-space nor an r-space</u>.

Let A be an uncountable set with the discrete topology and let X be the quotient space of the real numbers R obtained by identifying the integers Z . Then X is an $\aleph_0$-space (hence paracompact and perfectly normal) since the projection $p : R \twoheadrightarrow X$ is closed.

X is not an r-space. Let us denote the point of X which is the image under p of the integers by z. If $\{U_n(z)\}_{n-1}^{\infty}$ is any sequence of neighborhoods of z, then each $U_n(z)$ contains a neighborhood $V_n(m)$ (in R) of each integer m. Clearly, the family $\{V_n(m)$: m is an integer$\}$ can be taken to be pairwise disjoint for each $n \in N$. Then if one chooses $x_n \in V_n(n) - \{n\}$, it is apparent that $F = \{x_1, x_2, \ldots\}$ does not have compact closure in X. For $W(z) = \bigcup_{n>o} (V_n(n) - \{x_n\}) \cup \bigcup_{m \leq o} V_n(m)$ is a neighborhood of z in X which does not meet F. Hence F is closed in X and $\{V_n(n) - \{n\}\}$ is an open cover of F in X which has no finite subcover.

Let $X' = A \times X$. Then X' is an $\aleph_\sigma$-space by 6.2.4 since the family $\{\{a\} \times X : a \in A\}$ is locally finite and closed. X' is paracompact [4, p. 837, Prop. 5], and is clearly neither an $\aleph_o$-space nor an r-space.

At the end of Chapter 7 we present an example of a regular, T_1 space which is not an $\aleph_\sigma$-space but which is a continuous open image of an $\aleph_o$-space.

CHAPTER 7

k-Spaces

7.1 k-Spaces

A Hausdorff space X is called a k-space if it has the weak topology determined by the family of its compact subsets. That is, U is open in X iff $U \cap C$ is open in C for each compact $C \subset X$.

If (X,τ) is a Hausdorff space, the k-extension of τ is the family τ_k of all subsets $U \subset X$ such that $U \cap C$ is open in C for each compact $C \subset X$. (X,τ_k) is denoted by $k(X)$. It is well known [3, p. 241, K] that τ and τ_k yeild the same compact subsets of X.

In this chapter we prove that $k(X)$ is an $\aleph_\sigma$-space if X is an $\aleph_\sigma$-space. Actually, as for $\aleph_0$-spaces, a stronger statement is true. We shall need the following lemmas, one of which (lemma 7.1.2) is Lemma 8.1 of [5].

<u>Lemma 7.1.1</u>. <u>If T_1 and T_2 are two Hausdorff topologies for a set X yeilding the same compact subsets, then for each compact $C \subset X$, $U \in T_2$ (T_1) implies that $U \cap C$ is T_1 (T_2)-relatively open in C.</u>

<u>Proof</u>. Let $U \in T_2$ and let C be compact in X. Then $U \cap C$ is T_2-relatively open in C and therefore, $C-U$ is T_2-relatively closed

in C . C-U is then T_2 compact and thus C-U is T_1 compact. Then C-U is T_1-relatively closed in C so that $U \cap C$ is T_1-relatively open in C . We can now interchange T_1 and T_2 throughout the preceding argument and that completes the proof.

Lemma 7.1.2. *If C is a compact subset of a space X and if $S = \{x_n : n \in N\}$ is a sequence such that S is eventually in each neighborhood of C , then $K = C \cup S$ is compact.*

Proof. Let $\mathcal{U}$ be an open cover of K and let $V = \cup\{U_i : i \leq n , U_i \in \mathcal{U}\}$, where $C \subset V$. Then all but finitely many x_n's are in V . Thus K can be covered by finitely many members of $\mathcal{U}$ and that completes the proof.

Theorem 7.1.3. *If T_1 and T_2 are two Hausdorff topologies for a set X yeilding the same compact subsets, then (X,T_1) has a σ-locally finite generator iff (X,T_2) does.*

Proof. Suppose that (X,T_1) has a σ-locally finite generator $\mathcal{J} = \cup\{\mathcal{J}_n : n \in N\}$. Let $C \subset U \in T_2$, where C is compact. Let $\mathcal{J}(C) = \{T_n : n \in N\}$ be an enumeration of those members of $\mathcal{J}$ which meet C , and let $\{R_m : m \in N\}$ be the class of all finite unions of members of $\mathcal{J}(C)$ such that $C \subset R_m$ for each $m \in N$. For each $k \in N$, let $R'_k = \bigcap_{m=1}^{k} R_m$. Then for some $k \in N$, $C \subset R'_k \subset U$. Suppose not. Then for each $k \in N$, pick $x_k \in R'_k - U$. If V is any T_1-open

neighborhood of C, then the R_k''s are eventually subsets of V since $\mathfrak{J}$ is a generator for (X,T_1). By Lemma 7.1.2, $K = C \cup \{x_1, x_2, \ldots\}$ is compact. By Lemma 7.1.1, $U \cap K = C$ is T_1-relatively open in K. Then $B = \{x_1, x_2, \ldots\}$ is T_1-closed in K and hence, B is compact. Since $C \cap B = \emptyset$ and since B is T_1-closed, there is a neighborhood W of C such that $W \cap B = \emptyset$. But this is a contradiction, and thus $C \subset R_k' \subset U$ for some $k \in N$.

Now if $\hat{\mathfrak{J}}$ is the class of all finite intersections of members of $\mathfrak{J}$, then $\hat{\mathfrak{J}}$ is a σ-locally finite generator for (X,T_2). For we have shown in the proof of Lemma 5.1 that $\hat{\mathfrak{J}}$ is σ-locally finite and that R_k' can be expressed as a finite union of members of $\hat{\mathfrak{J}}$.

Corollary 7.1.4. A Hausdorff space X has a σ-locally finite generator *iff* $k(X)$ *does*.

Consequently, $k(X)$ *is an* $\aleph_\sigma$-*space whenever* X *is an* $\aleph_\sigma$-*space*.

7.2 Examples

In [5, Example 12.5], Michael presents an example of an $\aleph_\sigma$-space which is not a k-space. We shall reproduce this and another example of Michael's in order to show that there is a regular, T_1, non-$\aleph_\sigma$-space which is a continuous open image of an $\aleph_0$-space.

Example 7.2.1. Let $\beta(X)$ be the Stone-Cech compactification of a Tychonoff space X. Let $p \in \beta(N)-N$ and let $P = N \cup \{p\}$. Then every compact subset of P is finite, so P is an $\aleph_0$-space but not a k-space.

Proof. If $C \subset P$ were compact and infinite, then $C = (C \cap N)^-$, so C would be the closure of $C \cap N$ in $\beta(N)$. Hence C would be homeomorphic to $\beta(C \cap N)$ and thus to $\beta(N)$ which is impossible since C is countable and $\beta(N)$ is not.

Now the finite subsets of P form a pseudobase for P, and thus P is an $\aleph_0$-space. If P were a k-space, then $\{p\}$ would have to be open which it is not, and that completes the proof.

Example 7.2.2. A regular, T_1, non-$\aleph_\sigma$-space which is a continuous open image of an $\aleph_0$-space.

The space X of Example 4.3.2 is such a space. Since this X is not an $\aleph_\sigma$-space, it suffices to define an open mapping f from an $\aleph_0$-space X' onto X.

According to Example 4.3.2, X has a closed subspace A such that A and $X-A$ are both separable metrizable. Let $P = N \cup \{p\}$ be the space of Example 7.2.1, with N dense in P, such that all compact subsets of P are finite. Define $X' \subset X \times P$ by $X' = ((X-A) \times N) \cup (A \times \{p\})$. Let $p : X \times P \twoheadrightarrow X$ be the projection and let $f = p|X'$. Since f is continuous, we need only show that f is open and that X' is an $\aleph_0$-space.

Let U be open in X'. Then $U = V \cap X$, where V is open in $X \times P$. Now f maps $f^{-1}(A) = A \times \{p\}$ homeomorphically onto A. Hence $f(U) \cap A$ is relatively open in A. Then $B = A - (f(U) \cap A)$ is closed and hence $p^{-1}(B)$ is closed in $X \times P$. Now $p^{-1}(B) \cap U = \emptyset$ so that if $W = V - p^{-1}(B)$, then W is open in $X \times P$ and $W \cap X' = U$. Now if $y \in X$, then $p^{-1}(y)$ meets W iff it meets U. For if $y \in A$, this follows from the definition of W. If $y \in X\text{-}A$, then $X' \cap p^{-1}(y)$ is dense in $p^{-1}(y)$, and therefore (intersecting X' and $p^{-1}(y)$ with W) $U \cap p^{-1}(y)$ is dense in $W \cap p^{-1}(y)$. Thus $p(W) = p(U)$ and since p is an open mapping, $p(W) = p(U) = f(U)$ is open in X.

To show that X' is an $\aleph_0$-space, note first that it is regular and T_1 along with P and X. Moreover, X' is covered by the separable metric spaces $A \times \{p\}$ and $(X\text{-}A) \times N$. Then by [5, Proposition 7.7], it suffices to show that if $C \subset X'$ is compact, and if $C_1 = C \cap (A \times \{p\})$ and $C_2 = C \cap ((X\text{-}A) \times N)$, then C_1 and C_2 are both compact. C_1 is compact since $A \times \{p\}$ is closed in X'. To show that C_2 is compact, consider the projection $p_2 : X \times P \twoheadrightarrow P$. $p_2(C)$ is compact in P and hence, $p_2(C)$ is finite. Then $p_2(C) \cap N$ is closed in P. Now $C_2 = C \cap p_2^{-1}(p_2(C) \cap N)$ so that C_2 is compact.

CHAPTER 8

Function Spaces

8.1 The Compact-Open Topology for $\mathcal{C}(X,Y)$

If X and Y are spaces, Y^X denotes the family of all functions from X to Y. If $A \subset X$ and $B \subset Y$, then $W(A,B) = \{f \in Y^X : f(A) \subset B\}$. The compact-open topology for Y^X is that having sub-base the family of all sets of the form $W(C,U)$, with C compact in X and U open in Y. $\mathcal{C}(X,Y)$ will denote the space of continuous functions from X to Y with the compact-open topology.

If $K \subset \mathcal{C}(X,Y)$ and $A \subset X$, then $K(A) = \{f(x) : f \in K$ and $x \in A\}$. We denote by $\mathcal{K}(Y)$ the class of all non-empty compact subsets of Y. Then a function $F : X \to \mathcal{K}(Y)$ is called upper semi-continuous if $\{x \in X : F(x) \subset V\}$ is open in X for every open V in Y.

It is known [3, p. 223, Theorem 5] that the compact-open topology for $\mathcal{C}(X,Y)$ is jointly continuous on compacta whenever X is regular; that is, the map $P : \mathcal{C}(X,Y) \times X \to Y$ defined by $P(f,x) = f(x)$ is continuous on $\mathcal{C}(X,Y) \times C$ for each compact $C \subset X$. It follows that if $K \subset \mathcal{C}(X,Y)$ is compact and $C \subset X$ is compact, then $K(C)$ is compact.

A subset $K \subset \mathcal{C}(X,Y)$ is said to be evenly continuous iff

for each $x \in X$, each $y \in Y$, and each neighborhood U of y, there is a neighborhood V of x and a neighborhood W of y such that $f(V) \subset U$ whenever $f \in K$ and $f(x) \in W$.

In [5], Michael shows (Lemma 9.2) that if X is a k-space and Y is regular and $K \subset \mathcal{C}(X,Y)$ is compact, then the function $\phi : X \to \mathcal{K}(Y)$ defined by $\phi(x) = K(x)$ is upper semi-continuous. Since our theorem will require that X be a locally compact, regular space, we include another proof here, using this added restriction on X.

<u>Lemma 8.1.1</u>. <u>Let</u> X <u>be a locally compact, regular space and let</u> Y <u>be a regular Hausdorff space. Then if</u> $K \subset \mathcal{C}(X,Y)$ <u>is compact, the function</u> $\phi : X \to \mathcal{K}(Y)$ <u>defined by</u> $\phi(x) = K(x)$ <u>is upper semi-continuous</u>.

<u>Proof</u>. By Ascoli's theorem [3, p. 236, 21], K is evenly continuous. Let $x_1 \in \{x : \phi(x) \subset U\}$ for an open $U \subset Y$. $K(x_1)$, being compact, can be covered by a finite collection of open sets W_i, $i = 1,2,\ldots,n$ such that there are neighborhoods V_i, $i = 1,2,\ldots,n$ of x_1 such that $f(V_i) \subset U$ whenever $f(x_1) \in W_i$ and $f \in K$. Then if $N(x_1) = \bigcap_{i=1}^{n} V_i$, $N(x_1)$ is a neighborhood of x_1 in $\{x : \phi(x) \subset U\}$ and that completes the proof.

Recall that if X and Y are $\aleph_0$-spaces, then so is $\mathcal{C}(X,Y)$ [5, J]. The corresponding statement for $\aleph_\sigma$-spaces does not always hold as is shown in Example 8.3.1. We can however establish a modified version of this property for $\aleph_\sigma$-spaces. It contains a proof of Theorem 5.2 in

the special case where all the X_n's are the same space, since a countable set with the discrete topology is obviously a locally compact $\aleph_0$-space.

Before proceeding with the theorem, let us observe that if we call a class of subsets of a space compact whenever each member of the class is compact, then a locally compact $\aleph_0$-space has a countable, compact pseudobase. For if X is a locally compact $\aleph_0$-space and $\mathcal{P}$ is a countable pseudobase for X, then $\mathcal{P}' = \{P \in \mathcal{P} : P$ is compact$\}$ is also a pseudobase for X. This follows from the fact that if $C \subset U$, where C is compact and U is open in X, then there is a compact F such that $C \subset F^o \subset F \subset U$.

<u>Theorem</u> <u>8.1.2</u>. If X is a locally compact $\aleph_0$-space and Y is an $\aleph_\sigma$-space, then $\mathcal{C}(X,Y)$ is an $\aleph_\sigma$-space.

<u>Proof</u>. $\mathcal{C}(X,Y)$ is regular and T_1 since Y is so [3, p. 222,4].

Let $\mathcal{P} = \{P_n : n \in N\}$ be a compact pseudobase for X. Let $\mathcal{J} = \cup\{\mathcal{J}_n : n \in N\}$ be a σ-locally finite generator for Y. Let M be the class of all finite subsets of N and for each $m \in M$, let $\mathcal{J}'_m = \cup\{\mathcal{J}_n : n \in m\}$. As we have seen before, each $\mathcal{J}'_m$ is locally finite. For each $m \in M$, let $\mathcal{R}_m$ be the class of all finite unions of members of $\mathcal{J}'_m$.

If $P \in \mathcal{P}$, and $R = \bigcup_{k=1}^{r} T_k \in \mathcal{R}_m$, and $f \in \mathcal{C}(X,Y)$; let us say that $f(P) < R$ means that $f(P) \subset R$ and $f(P) \cap T_k \neq \emptyset$, $k = 1,2,\ldots,r$. Finally, let $\mathcal{F}_{i,m}$ be the class of all sets of the form

$W_*(P_i,R) = \{f \in \mathcal{C}(X,Y) : f(P_i) < R\ ,\ P_i \in \mathcal{P}$ and $R \in \mathcal{R}_m\}$. If $N_m(f(P_i))$ is a neighborhood of the compact set $f(P_i)$ which meets only finitely many $T \in \mathcal{J}'_m$, then $W(P_i\ , N_m(f(P_i)))$ is a neighborhood of f which meets only finitely many members of $\mathcal{F}_{i,m}$. For if $g \in W(P_i, N_m(f(P_i))) \cap W_*(P_i,R)$ for $R \in \mathcal{R}_m$, then $g(P_i) \subset R = \bigcup_{k=1}^{r} T_k \in \mathcal{R}_m$ and each $T_k \cap g(P_i) \neq \emptyset$. Since $g(P_i) \subset N_m(f(P_i))$ which meets only finitely many $T \in \mathcal{J}'_m$, it follows that there are only finitely many R's in $\mathcal{R}_m$ for which $W(P_i\ , N_m(f(P_i))) \cap W_*(P_i,R) \neq \emptyset$. Thus $\mathcal{F}_{i,m}$ is locally finite for each pair $(i,m) \in N \times M$.

Now let $\mathcal{S}$ be the sub-base for $\mathcal{C}(X,Y)$ consisting of all sets of the form $W(C,U)$, with C compact in X and U open in Y . Suppose that $K \subset W(C,U) \in \mathcal{S}$ for a compact K . By Lemma 8.1.1, $V = \{x : K(x) \subset U\}$ is open. Since $C \subset V$ and X is locally compact, there is a compact F such that $C \subset F^o \subset F \subset V$. Hence there is a $P_i \in \mathcal{P}$ such that $C \subset P_i \subset F^o \subset V$. Now $K(P_i) \subset K(V) \subset U$ and $K(P_i)$ is compact. Then there is an $m \in M$ such that $K(P_i) \subset R = \bigcup_{k=1}^{r} T_k \subset U$, with $R \in \mathcal{R}_m$. For each $f \in K$, let $R_f = \cup\{T_k \in \{T_k\}_{k=1}^{r} : f(P_i) \cap T_k \neq \emptyset\}$. Clearly, there are only finitely many distinct R_f's . It is also clear that $f(P_i) < R_f$ for each $f \in K$. Then $K \subset \cup\{W_*(P_i, R_f) : f \in K\} \subset W(P_i,R) \subset W(C,U)$. Thus $\mathcal{F} = \cup\{\mathcal{F}_{i,m} : (i,m) \in N \times M\}$ is a σ-locally finite class which generates an $\mathcal{S}$-pseudobase for $\mathcal{C}(X,Y)$. The proposition now follows from Lemma 5.1.

The hypothesis that X be a locally compact $\aleph_0$-space in 8.1.2 implies that X is separable metrizable [5, C]. I have not been able

to prove or disprove that a separable metric space is enough in 8.1.2. I conjecture that it is not.

8.2 A Metrizable Uniform Space $(Y^X, \mathcal{U})$

We shall now construct a uniformity $\mathcal{U}$ for the set Y^X, where Y is a paracompact $\aleph_\sigma$-space and X is an arbitrary set, such that $(Y^X, \mathcal{U})$ is metrizable. Let us first recall some definitions.

A cover $\mathcal{U}$ of a space X is said to be even if there is a neighborhood V of the diagonal $\Delta(X) = \{(x,x) : x \in X\}$ in $X \times X$ such that the family of all sets of the form $V|x| = \{y : (x,y) \in V\}$ refines $\mathcal{U}$. A neighborhood V of $\Delta(X)$ is such to be symetric if $V = V^{-1} = \{(x,y) : (y,x) \in V\}$. If U and V are subsets of $X \times X$, then $U \circ V = \{(x,z) :$ there is $y \in X$ such that $(x,y) \in V$ and $(y,z) \in U\}$.

A uniformity for a set X is a non-void family $\mathcal{U}$ of subsets of $X \times X$ having the following properties:

(a) $\Delta(X) \subset U$ for each $U \in \mathcal{U}$.

(b) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$.

(c) If $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in \mathcal{U}$.

(d) If $U \in \mathcal{U}$ and $V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.

(e) If $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

The pair $(X,\mathcal{U})$ is called a uniform space. The topology $\tau(\mathcal{U})$ for X induced by the uniformity $\mathcal{U}$ (the uniform topology) is the family of all subsets W of X such that for each $x \in W$, there is a $U \in \mathcal{U}$ such that $U|x| \subset W$.

A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ for a set X is called a base for $\mathcal{U}$ if each $U \in \mathcal{U}$ contains some $B \in \mathcal{B}$. By [3, p. 177, 2], $\mathcal{B}$ is a base for some uniformity for X if $\mathcal{B}$ is a non-void family of subsets of $X \times X$ satisfying (a) through (d) above, with $\mathcal{U}$ replaced by $\mathcal{B}$.

Each metric ρ for a set X generates a uniformity in the following way: For each positive number r, let $V_{\rho,r} = \{(x,y) : \rho(x,y) < r\}$. Then:

(a) $\Delta(X) \subset V_{\rho,r}$ for each $r > 0$.

(b) $V_{\rho,r} = V_{\rho,r}$.

(c) $V_{\rho,r} \cap V_{\rho,s} = V_{\rho,t}$, where $t = \min\{r,s\}$.

(d) $V_{\rho,r} \circ V_{\rho,r} \subset V_{\rho,2r}$.

Thus the family of all sets of the form $V_{\rho,r}$ is a base for a uniformity for X. A uniform space $(X,\mathcal{U})$ is said to be metrizable iff it is possible to introduce a metric ρ for X such that $\mathcal{U}$ is the uniformity generated by ρ.

<u>Theorem 8.2.1</u>. <u>If</u> X <u>is a set and</u> Y <u>is a paracompact</u> $\aleph_\sigma$<u>-space, there is a uniformity</u> $\mathcal{U}$ <u>for</u> Y^X <u>such that</u> $(Y^X, \mathcal{U})$ <u>is metrizable and, provided that</u> Y <u>is not discrete and</u> $f(X) = Y$ <u>for some</u> $f \in Y^X$, <u>the uniform topology</u> $\tau(\mathcal{U})$ <u>for</u> Y^X <u>is not discrete</u>.

<u>Proof</u>. Let $\mathcal{J} = \cup\{\mathcal{J}_m : m \in N\}$ be a closed σ-locally finite generator for Y and for each $n \in N$, let $\mathcal{J}_n' = \cup\{\mathcal{J}_m : m \leq n\}$. For each $y \in Y$ and for each $n \in N$, let $U_n(y) = Y - \cup\{T \in \mathcal{J}_n' : y \notin T\}$. Then each $U_n(y)$ is an open neighborhood of y.

If y_1 and y_2 are distinct points of Y, then since Y is Hausdorff, there are disjoint members T_1 and T_2 of $\mathcal{J}$ such that $y_1 \in T_1$ and $y_2 \in T_2$. Hence there are integers n_1 and n_2 such that $y_1 \notin U_{n_2}(y_2)$ and $y_2 \notin U_{n_1}(y_1)$.

Since Y is paracompact, each open cover of Y is even [3, p. 155, 27]. Furthermore if U is a neighborhood of $\Delta(Y)$ then there is a symmetric neighborhood V of $\Delta(Y)$ such that $V \circ V \subset U$ [3, p. 157, 30]. Then for each $n \in N$, let $\mathcal{U}_n = \{U_n(y) ; y \in Y\}$. Let V_1 be a symmetric neighborhood of $\Delta(Y)$ such that $\{V_1|y| : y \in Y\}$ refines $\mathcal{U}_1$. Let W_2 be a symmetric neighborhood of $\Delta(Y)$ such that $\{W_2|y| : y \in Y\}$ refines $\mathcal{U}_2$ and let $\tilde{W}_2$ be a symmetric neighborhood of $\Delta(Y)$ such that $\tilde{W}_2 \circ \tilde{W}_2 \subset V_1$. Let $V_2 = W_2 \cap \tilde{W}_2$. Then V_2 is a symmetric neighborhood of $\Delta(Y)$ and $V_2 \circ V_2 \subset V_1$. Inductively, define V_{n+1} to be a symmetric neighborhood of $\Delta(Y)$ such that $\{V_{n+1}|y| : y \in Y\}$ refines $\mathcal{U}_{n+1}$ and $V_{n+1} \circ V_{n+1} \subset V_n$.

Let Z be the set $Y^X \times Y^X$. For each $n \in N$, let $U_n = \{(f,g) \in Z : (f(x), g(x)) \in V_n$ for each $x \in X\}$. Then $\mathcal{B} = \{U_n : n \in N\}$ is a base for a uniformity $\mathcal{U}$ for Y^X. We shall show that $\mathcal{B}$ satisifes the conditions (a) through (d) of [3, p. 177, 2].

(a) $\Delta(Y^X) \subset U_n$ for each $n \in N$. For if $(f,f) \in \Delta(Y^X)$, then $(f(x), f(x)) \in \Delta(Y) \subset V_n$ for each $x \in X$ and for each $n \in N$

(b) If $U_n \in \mathcal{B}$, then $U_n^{-1} = U_n$. For $\{(f,g) : (f(x), g(x)) \in V_n$ for each $x \in X\} = \{g,f) : (g(x), f(x)) \in V_n$ for each $x \in X\}$ since $V_n = V_n^{-1}$.

(c) $U_n \circ U_n \subset U_{n-1}$ for each $n \in N$. For if $(f,g) \in U_n \circ U_n$, then there is an $h \in Y^X$ such that $(f,h) \in U_n$ and $(h,g) \in U_n$. Then $(f(x), h(x)) \in V_n$ and $(h(x), g(x)) \in V_n$ for each $x \in X$. It follows that $(f(x), g(x)) \in V_n \circ V_n \subset V_{n-1}$ for each $x \in X$ and therefore, that $U_n \circ U_n \subset U_{n-1}$.

(d) $U_n \cap U_m = U_m$ if $m \geq n$. Let $(f,g) \in U_m$. Then $(f(x), g(x)) \in V_m$ for each $x \in X$ and hence, $(f(x), g(x)) \in V_{m-1}$ for each $x \in X$ since $V_m \subset V_m \circ V_m \subset V_{m-1}$.

We now show that the uniformity $\mathcal{U}$ for Y^X having base $\mathcal{B}$ induces a T_1 topology on Y^X. Let f and g be distinct members of Y^X. Then there is some $x \in X$ such that $f(x) \neq g(x)$. Let n be the first integer such that there are disjoint members T_1 and T_2 of $\mathcal{T}_n'$, with $f(x) \in T_1$ and $g(x) \in T_2$. If $f(x) \in V_n|g(x)| \subset U_n(y)$ for some $y \in Y$, then $y \in T_1 \cap T_2$ by definition of $U_n(y)$, which

is a contradiction. Similarly, $g(x) \notin V_n|f(x)|$. Now it follows that $f \notin U_n|g|$ and $g \notin U_n|f|$.

Then $(Y^X, \mathcal{U})$ is a Hausdorff uniform space [3, p. 180] and hence, is metrizable [3, p. 186, 13].

Finally, suppose that $\{f\} = U_n|f|$ for some $f \in Y^X$ such that $f(X) = Y$. Then $\{f\} = \{g : (f,g) \in U_n\}$. Now $U_n = \{(g,h) : (g(x), h(x)) \in V_n$ for each $x \in X\}$ and V_n is a neighborhood of $\Delta(Y)$. Thus if f is the only member of $U_n|f|$, then $V_n = \Delta(Y)$ and consequently, Y is discrete. For suppose not. Let $(y_1, y_2) = V_n - \Delta(Y)$ where $y_2 = f(x_1)$ for $x_1 \in X$. Since $y_1 \neq y_2$, there is a $g \in Y^X$ such that $g \neq f$ and $(f(x), g(x)) \in V_n$ for each $x \in X$. Merely take $g \in Y^X$ to be that function which is equal to $f(x)$ for each $x \in X - \{x_1\}$ and let $g(x_1) = y_2$. But this contradicts $\{f\} = U_n|f|$. Then, unless Y is discrete (some $V_n = \Delta(Y)$) , the points $f \in Y^X$ such that $f(X) = Y$ in the uniform topology on Y^X are not open and that completes the proof.

8.3 Examples

X cannot be replaced by an arbitrary metric space in 8.1.2 as the following example shows.

Example 8.3.1 . Let X be the closed unit interval with the discrete topology and let Y be the same set with the usual topology. Then

$\mathcal{C}(X,Y) = Y^X$ __is compact__, __but is not an__ $\aleph_\sigma$-__space__.

__Proof.__ If Y^X were an $\aleph_\sigma$-space, then by Corollary 4.2.2, it would be metrizable which it is not.

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